

# Convex Relaxation of Optimal Power Flow—Part II: Exactness

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**Abstract**—This tutorial summarizes recent advances in the convex relaxation of the optimal power flow (OPF) problem, focusing on structural properties rather than algorithms. Part I presents two power flow models, formulates OPF and their relaxations in each model, and proves equivalence relations among them. Part II presents sufficient conditions under which the convex relaxations are exact.

**Index Terms**—Convex relaxation, optimal power flow, power systems, quadratically constrained quadratic program (QCQP), second-order cone program (SOCP), semidefinite program (SDP), semidefinite relaxation.

## I. INTRODUCTION

THE OPTIMAL power flow (OPF) problem is fundamental in power systems since it underlies many applications, such as economic dispatch, unit commitment, state estimation, stability and reliability assessment, volt/var control, demand response, etc. OPF seeks to optimize a certain objective function, such as power loss, generation cost, and/or user utilities, subject to Kirchhoff's laws as well as capacity, stability, and security constraints on the voltages and power flows. There has been a great deal of research on OPF since Carpentier's first formulation in 1962 [1]. Recent surveys can be found in, e.g., [2]–[13].

OPF is generally nonconvex and NP-hard, and a large number of optimization algorithms and relaxations have been proposed. To the best of our knowledge solving OPF through semidefinite relaxation is first proposed in [14] as a second-order cone program (SOCP) for radial (tree) networks and in [15] as a semidefinite program (SDP) for general networks in a bus injection model. It is first proposed in [16] and [17] as an SOCP for radial networks in the branch flow model of [18] and [19]. While these convex relaxations have been illustrated numerically in [14] and [15], whether or when they will turn out to be exact is first studied in [20]. Exploiting graph sparsity

to simplify the SDP relaxation of OPF is first proposed in [21] and [22], and analyzed in [23] and [24].

Solving OPF through convex relaxation offers several advantages, as discussed in Part I of this tutorial [25, Sec. I]. In particular, it provides the ability to check whether a solution is globally optimal. If it is not, the solution provides a lower bound on the minimum cost and, hence, a bound on how far any feasible solution is from optimality. Unlike approximations, if a relaxed problem is infeasible, it is a certificate that the original OPF is infeasible.

This tutorial presents main results on convex relaxations of OPF developed in the last few years. In Part I [25], we present the bus injection model (BIM) and the branch flow model (BFM), formulate OPF within each model, and prove their equivalence. The complexity of OPF formulated here lies in the quadratic nature of power flows, i.e., the nonconvex quadratic constraints on the feasible set of OPF. We characterize these feasible sets and design convex supersets that lead to three different convex relaxations based on semidefinite programming, chordal extension, and second-order cone programming. When a convex relaxation is exact, an optimal solution of the original nonconvex OPF can be recovered from every optimal solution of the relaxation. In Part II, we summarize main sufficient conditions that guarantee the exactness of these relaxations.

Network topology turns out to play a critical role in determining whether a relaxation is exact. In Section II, we review the definitions of OPF and their convex relaxations developed in [25]. We also define the notion of exactness adopted in this paper. In Section III, we present three types of sufficient conditions for these relaxations to be exact for radial networks. These conditions are generally not necessary and they have implications on allowable power injections, voltage magnitudes, or voltage angles as follows.

- A) *Power injections*: These conditions require that not both constraints on real and reactive power injections be binding at both ends of a line.
- B) *Voltages magnitudes*: These conditions require that the upper bounds on voltage magnitudes not be binding. They can be enforced through affine constraints on power injections.
- C) *Voltage angles*: These conditions require that the voltage angles across each line be sufficiently close. This is needed also for stability reasons.

These conditions and their references are summarized in Tables I and II. Some of these sufficient conditions are proved using BIM and others using BFM. Since these two models are equivalent (in the sense that there is a linear bijection between their solution sets [24], [25]), these sufficient conditions apply

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TABLE I  
SUFFICIENT CONDITIONS FOR RADIAL (TREE) NETWORKS

type	condition	model	reference	remark
A	power injections	BIM, BFM	[26], [27], [28], [29], [30] [31], [16], [17]	
B	voltage magnitudes	BFM	[32], [33], [34], [35]	allows general injection region
C	voltage angles	BIM	[36], [37]	makes use of branch power flows

TABLE II  
SUFFICIENT CONDITIONS FOR MESH NETWORKS

network	condition	reference	remark
with phase shifters	type A, B, C	[17, Part II], [38]	equivalent to radial networks
direct current	type A	[17, Part I], [20], [39]	assumes nonnegative voltages
	type B	[40], [41]	assumes nonnegative voltages

to both models. The proofs of these conditions typically do not require that the cost function be convex (they focus on the feasible sets and usually only need the cost function to be monotonic). Convexity is required, however, for efficient computation. Moreover, it is proved in [35] using BFM that when the cost function is convex, then exactness of the SOCP relaxation implies uniqueness of the optimal solution for radial networks. Hence, the equivalence of BIM and BFM implies that any of the three types of sufficient conditions guarantees that, for a radial network with a convex cost function, there is a unique optimal solution and it can be computed by solving an SOCP. Since the SDP and chordal relaxations are equivalent to the SOCP relaxation for radial networks [24], [25], these results apply to all three types of relaxations. Empirical evidence suggests that some of these conditions are likely satisfied in practice. This is important as most power distribution systems are radial.

These conditions are insufficient for general mesh networks because they cannot guarantee that an optimal solution of a relaxation satisfies the cycle condition discussed in [25]. In Section IV, we show that these conditions are, however, sufficient for mesh networks that have tunable phase shifters at strategic locations. The phase shifters effectively make a mesh network behave like a radial network as far as convex relaxation is concerned. The result can help determine whether a network with a given set of phase shifters can be convexified and, if not, where additional phase shifters are needed for convexification. These conditions are also sufficient for direct current (dc) mesh networks where all variables are in the real rather than complex domain. Counterexamples are known where SDP relaxation is not exact, especially for ac mesh networks without tunable phase shifters [42]–[44]. We discuss three recent approaches for global optimization of OPF when the relaxations discussed in this tutorial fail.

We conclude in Section V. All proofs can be found in the original papers or the arXiv version of this paper.

## II. OPF AND ITS RELAXATIONS

We use the notations and definitions from Part I of this paper. In this section, we summarize the OPF problems and their relaxations developed there; see [25] for details.

We adopt in this paper a strong sense of “exactness” where we require the optimal solution set of the OPF problem and that of its relaxation to be equivalent. This implies that an optimal solution of the nonconvex OPF problem can be recovered from *every* optimal solution of its relaxation. This is important because it ensures any algorithm that solves an exact relaxation always produces a globally optimal solution to the OPF problem. Indeed, interior point methods for solving SDPs tend to produce a solution matrix with a maximum rank [45], so can miss a rank-1 solution if the relaxation has non-rank-1 solutions as well. It can be difficult to recover an optimal solution of OPF from such a non-rank-1 solution, and our definition of exactness avoids this complication. See Section II-C for detailed justifications.

### A. Bus Injection Model

The BIM adopts an undirected graph  $G^1$  and can be formulated in terms of just the complex voltage vector  $V \in \mathbb{C}^{n+1}$ . The feasible set is described by the following constraints:

$$\underline{s}_j \leq \sum_{k:(j,k) \in E} y_{jk}^H V_j (V_j^H - V_k^H) \leq \bar{s}_j, \quad j \in N^+ \quad (1a)$$

$$\underline{v}_j \leq |V_j|^2 \leq \bar{v}_j, \quad j \in N^+ \quad (1b)$$

where  $\underline{s}_j, \bar{s}_j, \underline{v}_j, \bar{v}_j$ , possibly  $\pm\infty \pm i\infty$ , are given bounds on power injections and voltage magnitudes. Note that the vector  $V$  includes  $V_0$  which is assumed given ( $\underline{v}_0 = \bar{v}_0$  and  $\angle V_0 = 0^\circ$ ) unless otherwise specified.

The problem of interest is

OPF :

$$\min_{V \in \mathbb{C}^{n+1}} C(V) \text{ subject to } V \text{ satisfies (1).} \quad (2)$$

For relaxations, consider the partial matrix  $W_G$  defined on the network graph  $G$  that satisfies

$$\underline{s}_j \leq \sum_{k:(j,k) \in E} y_{jk}^H ([W_G]_{jj} - [W_G]_{jk}) \leq \bar{s}_j, \quad j \in N^+ \quad (3a)$$

$$\underline{v}_j \leq [W_G]_{jj} \leq \bar{v}_j, \quad j \in N^+. \quad (3b)$$

<sup>1</sup>We will use “bus” and “node” interchangeably and “line” and “link” interchangeably.

We say that  $W_G$  satisfies the *cycle condition* if for every cycle  $c$  in  $G$

$$\sum_{(j,k) \in c} \angle[W_G]_{jk} = 0 \pmod{2\pi}. \quad (4)$$

We assume the cost function  $C$  depends on  $V$  only through  $VV^H$  and use the same symbol  $C$  to denote the cost in terms of a full or partial matrix. Moreover, we assume  $C$  depends on the matrix only through the submatrix  $W_G$  defined on the network graph  $G$ . See [25] for more details including the definitions of  $W_{c(G)} \succeq 0$  and  $W_G(j, k) \geq 0$ . Define the convex relaxations

*OPF-sdp* :

$$\min_{W \in \mathbb{S}^{n+1}} C(W_G) \quad \text{subject to } W_G \text{ satisfies (3), } W \succeq 0 \quad (5)$$

*OPF-ch* :

$$\min_{W_{c(G)}} C(W_G) \quad \text{subject to } W_G \text{ satisfies (3), } W_{c(G)} \succeq 0 \quad (6)$$

*OPF-socp* :

$$\begin{aligned} \min_{W_G} C(W_G) \quad & \text{subject to } W_G \text{ satisfies (3),} \\ & W_G(j, k) \geq 0, (j, k) \in E. \end{aligned} \quad (7)$$

For BIM, we say that OPF-sdp (5) is *exact* if every optimal solution  $W^{\text{sdp}}$  of OPF-sdp is psd rank-1; OPF-ch (6) is *exact* if every optimal solution  $W_{c(G)}^{\text{ch}}$  of OPF-ch is psd rank-1 (i.e., the principal submatrices  $W_{c(G)}^{\text{ch}}(q)$  of  $W_{c(G)}^{\text{ch}}$  are psd rank-1 for all maximal cliques  $q$  of the chordal extension  $c(G)$  of graph  $G$ ); OPF-socp (7) is *exact* if every optimal solution  $W_G^{\text{socp}}$  of OPF-socp is  $2 \times 2$  psd rank-1 and satisfies the cycle condition (4). To recover an optimal solution  $V^{\text{opt}}$  of OPF (2) from  $W^{\text{sdp}}$  or  $W_{c(G)}^{\text{ch}}$  or  $W_G^{\text{socp}}$ , see [25, Sec. IV-D].

## B. Branch Flow Model

The BFM adopts a directed graph  $\tilde{G}$  and is defined by the following set of equations:

$$\sum_{k:j \rightarrow k} S_{jk} = \sum_{i:i \rightarrow j} (S_{ij} - z_{ij}|I_{ij}|^2) + s_j, \quad j \in N^+ \quad (8a)$$

$$I_{jk} = y_{jk}(V_j - V_k), \quad j \rightarrow k \in \tilde{E} \quad (8b)$$

$$S_{jk} = V_j I_{jk}^H, \quad j \rightarrow k \in \tilde{E}. \quad (8c)$$

Denote the variables in BFM (8) by  $\tilde{x} := (S, I, V, s) \in \mathbb{C}^{2(m+n+1)}$ . Note that the vectors  $V$  and  $s$  include  $V_0$  (given) and  $s_0$ , respectively. Recall from [25] the variables  $x := (S, \ell, v, s) \in \mathbb{R}^{3(m+n+1)}$  that are related to  $\tilde{x}$  by the mapping  $x = h(\tilde{x})$  with  $\ell_{jk} := |I_{jk}|^2$  and  $v_j := |V_j|^2$ . The operational constraints are

$$\underline{v}_j \leq v_j \leq \bar{v}_j, \quad j \in N^+ \quad (9a)$$

$$\underline{s}_j \leq s_j \leq \bar{s}_j, \quad j \in N^+. \quad (9b)$$

We assume the cost function depends on  $\tilde{x}$  only through  $x = h(\tilde{x})$ . Then the problem in BFM is

*OPF* :

$$\min_{\tilde{x}} C(x) \quad \text{subject to } \tilde{x} \text{ satisfies (8), (9).} \quad (10)$$

For SOCP relaxation, consider

$$\sum_{k:j \rightarrow k} S_{jk} = \sum_{i:i \rightarrow j} (S_{ij} - z_{ij}\ell_{ij}) + s_j, \quad j \in N^+ \quad (11a)$$

$$v_j - v_k = 2\text{Re}(z_{jk}^H S_{jk}) - |z_{jk}|^2 \ell_{jk}, \quad j \rightarrow k \in \tilde{E} \quad (11b)$$

$$v_j \ell_{jk} \geq |S_{jk}|^2, \quad j \rightarrow k \in \tilde{E}. \quad (11c)$$

We say that  $x$  satisfies the *cycle condition* if

$$\exists \theta \in \mathbb{R}^n \text{ such that } B\theta = \beta(x) \pmod{2\pi} \quad (12)$$

where  $B$  is the  $m \times n$  reduced incidence matrix and, given  $x := (S, \ell, v, s)$ ,  $\beta_{jk}(x) := \angle(v_j - z_{jk}^H S_{jk})$  can be interpreted as the voltage angle difference across line  $j \rightarrow k$  implied by  $x$  (See [25, Sec. V]). The SOCP relaxation in BFM is

*OPF-socp* :

$$\min_x C(x) \quad \text{subject to } x \text{ satisfies (11), (9).} \quad (13)$$

For BFM, OPF-socp (13) in BFM is *exact* if every optimal solution  $x^{\text{socp}}$  attains equality in (11c) and satisfies the cycle condition (12). See [25, Sec. V-A] on how to recover an optimal solution  $\tilde{x}^{\text{opt}}$  of OPF (10) from any optimal solution  $x^{\text{socp}}$  of its SOCP relaxation.

## C. Exactness

The definition of exactness adopted in this paper is more stringent than needed. Consider SOCP relaxation in BIM as an illustration (the same applies to the other relaxations in BIM and BFM). For any sets  $A$  and  $B$ , we say that  $A$  is *equivalent* to  $B$ , denoted by  $A \equiv B$ , if there is a bijection between these two sets. Let  $\mathbb{M}(A)$  denote the set of minimizers when a certain function is minimized over  $A$ .

Let  $\mathbb{V}$  and  $\mathbb{W}_G^+$  denote the feasible sets of OPF (2) and OPF-socp (7), respectively:

$$\mathbb{V} := \{V \in \mathbb{C}^{n+1} | V \text{ satisfies (1)}\}$$

$$\mathbb{W}_G^+ := \{W_G | W_G \text{ satisfies (3), } W_G(j, k) \geq 0, (j, k) \in E\}.$$

Consider the following subset of  $\mathbb{W}_G^+$ :

$$\mathbb{W}_G := \{W_G | W_G \text{ satisfies (3), (4), } W_G(j, k) \geq 0,$$

$$\text{rank } W_G(j, k) = 1, (j, k) \in E\}.$$

Our definition of exact SOCP relaxation is that  $\mathbb{M}(\mathbb{W}_G^+) \subseteq \mathbb{W}_G$ . In particular, *all* optimal solutions of OPF-socp must be  $2 \times 2$  psd rank-1 and satisfy the cycle condition (4). Since  $\mathbb{W}_G \equiv \mathbb{V}$  (see [25]), exactness requires that the set of optimal solutions of OPF-socp (7) be equivalent to that of OPF (2), i.e.,  $\mathbb{M}(\mathbb{W}_G^+) = \mathbb{M}(\mathbb{W}_G) \equiv \mathbb{M}(\mathbb{V})$ .

If  $M(W_G^+) \supsetneq M(W_G) \equiv M(V)$ , then OPF-socp (7) is not exact according to our definition. Even in this case, however, every sufficient condition in this paper guarantees that an optimal solution of OPF can be easily recovered from an optimal solution of the relaxation that is outside  $W_G$ . The difference between  $M(W_G^+) = M(W_G)$  and  $M(W_G^+) \supsetneq M(W_G)$  is often minor, depending on the objective function; see Remarks 1 and 2 and comments after Theorems 5 and 8 in Section III. Hence, we adopt the more stringent definition of exactness for simplicity.

### III. RADIAL NETWORKS

In this section, we summarize the three types of sufficient conditions listed in Table I for semidefinite relaxations of OPF to be exact for radial (tree) networks. These results are important since most distribution systems are radial.

For radial networks, if SOCP relaxation is exact, then SDP and chordal relaxations are also exact (see [25, Theor. 5, 9]). We hence focus in this section on the exactness of OPF-socp in both BIM and BFM. Since the cycle conditions (4) and (12) are vacuous for radial networks, OPF-socp (7) is exact if all of its optimal solutions are  $2 \times 2$  rank-1 and OPF-socp (13) is exact if all of its optimal solutions attain equalities in (11c). We will freely use either BIM or BFM in discussing these results. To avoid triviality, we make the following assumption throughout this paper:

The voltage lower bounds satisfy  $\underline{v}_j > 0$ ,  $j \in N^+$ . The original problems OPF (2) and (10) are feasible.

#### A. Linear Separability

We will first present a general result on the exactness of the SOCP relaxation of general QCQP and then apply it to OPF. This result is first formulated and proved using a duality argument in [27], generalizing the result of [26]. It is proved using a simpler argument in [31].

Fix an undirected graph  $G = (N^+, E)$ , where  $N^+ := \{0, 1, \dots, n\}$  and  $E \subseteq N^+ \times N^+$ . Fix Hermitian matrices  $C_l \in \mathbb{S}^{n+1}$ ,  $l = 0, \dots, L$ , defined on  $G$ , i.e.,  $[C_l]_{jk} = 0$  if  $(j, k) \notin E$ . Consider QCQP

$$\min_{x \in \mathbb{C}^{n+1}} x^H C_0 x \quad (14a)$$

$$\text{subject to } x^H C_l x \leq b_l, \quad l = 1, \dots, L \quad (14b)$$

where  $C_0, C_l \in \mathbb{C}^{(n+1) \times (n+1)}$ ,  $b_l \in \mathbb{R}$ ,  $l = 1, \dots, L$ , and its SOCP relaxation where the optimization variable ranges over Hermitian partial matrices  $W_G$

$$\min_{W_G} \text{tr } C_0 W_G \quad (15a)$$

$$\text{subject to } \text{tr } C_l W_G \leq b_l, \quad l = 1, \dots, L \quad (15b)$$

$$W_G(j, k) \geq 0, \quad (j, k) \in E. \quad (15c)$$

The following result is proved in [27] and [31]. It can be regarded as an extension of [46] on the SOCP relaxation of

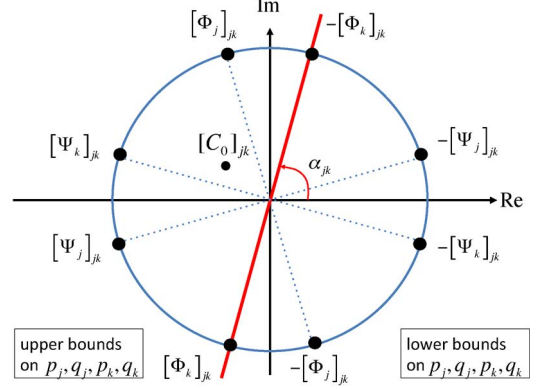


Fig. 1. Condition A2' on a line  $(j, k) \in E$ . The quantities  $([\Phi_j]_{jk}, [\Psi_j]_{jk}, [\Phi_k]_{jk}, [\Psi_k]_{jk})$  on the left-half plane correspond to finite upper bounds on  $(p_j, p_k, q_j, q_k)$  in (16a) and (16b);  $(-[\Phi_j]_{jk}, -[\Psi_j]_{jk}, -[\Phi_k]_{jk}, -[\Psi_k]_{jk})$  on the right-half plane correspond to finite lower bounds on  $(p_j, p_k, q_j, q_k)$ . A2' is satisfied if there is a line through the origin, specified by the angle  $\alpha_{jk}$ , so that the quantities corresponding to *finite* upper or lower bounds on  $(p_j, p_k, q_j, q_k)$  lie on one side of the line, possibly on the line itself. The load over-satisfaction condition in [26], [30] corresponds to the Im-axis that excludes all quantities on the right-half plane. The sufficient condition in [29, Theor. 2] corresponds to the red line in the figure that allows a finite lower bound on the real power at one end of the line, i.e.,  $p_j$  or  $p_k$  but not both, and no finite lower bounds on reactive powers  $q_j$  and  $q_k$ .

QCQP from the real domain to the complex domain. Consider (see Fig. 1 for an illustration)<sup>2</sup>:

A1: The cost matrix  $C_0$  is positive definite.

A2: For each link  $(j, k) \in E$ , there exists an  $\alpha_{jk}$  such that  $\angle[C_l]_{jk} \in [\alpha_{ij}, \alpha_{ij} + \pi]$  for all  $l = 0, \dots, L$ .

Let  $C^{\text{opt}}$  and  $C^{\text{socp}}$  denote the optimal values of QCQP (14) and SOCP (15), respectively.

**Theorem 1:** Suppose  $G$  is a tree and A2 holds. Then  $C^{\text{opt}} = C^{\text{socp}}$  and an optimal solution of QCQP (14) can be recovered from every optimal solution of SOCP (15).

**Remark 1:** The proof of Theorem 1 prescribes a simple procedure to recover an optimal solution of QCQP (14) from any optimal solution of its SOCP relaxation (15). The construction does not need the optimal solution of SOCP (15) to be  $2 \times 2$  rank-1. Hence, the SOCP relaxation may not be exact according to our definition of exactness, i.e., some optimal solutions of (15) may be  $2 \times 2$  psd but not  $2 \times 2$  rank-1. If the objective function is strictly convex, however, then the optimal solution sets of QCQP (14) and SOCP (15) are indeed equivalent.

**Corollary 2:** Suppose  $G$  is a tree and A1–A2 hold. Then SOCP (15) is exact.

We now apply Theorem 1 to our OPF problem in a standard form QCQP [27]

$$\begin{aligned} \min_{x \in \mathbb{C}^n} \quad & V^H C_0 V \\ \text{s.t.} \quad & V^H \Phi_j V \leq \bar{p}_j, \quad V^H (-\Phi_j) V \leq -\underline{p}_j \end{aligned} \quad (16a)$$

$$V^H \Psi_j V \leq \bar{q}_j, \quad V^H (-\Psi_j) V \leq -\underline{q}_j \quad (16b)$$

$$V^H J_j V \leq -\bar{v}_j, \quad V^H (-J_j) V \leq -\underline{v}_j$$

<sup>2</sup>All angles should be interpreted as “mod  $2\pi$ ”, i.e., projected onto  $(-\pi, \pi]$ .



for some Hermitian matrices  $C_0, \Phi_j, \Psi_j, J_j$  where  $j \in N^+$ . A2 depends only on the off-diagonal entries of  $C_0, \Phi_j, \Psi_j$  ( $J_j$  are diagonal matrices). It implies a simple pattern on the power injection constraints (16a), (16b). Let  $y_{jk} = g_{jk} - \mathbf{i}b_{jk}$  with  $g_{jk} > 0, b_{jk} > 0$ . Then we have (from [27])

$$[\Phi_k]_{ij} = \begin{cases} \frac{1}{2}Y_{ij} = -\frac{1}{2}(g_{ij} - \mathbf{i}b_{ij}) & \text{if } k = i \\ \frac{1}{2}Y_{ij}^H = -\frac{1}{2}(g_{ij} + \mathbf{i}b_{ij}) & \text{if } k = j \\ 0 & \text{if } k \notin \{i, j\} \end{cases}$$

$$[\Psi_k]_{ij} = \begin{cases} \frac{-1}{2\mathbf{i}}Y_{ij} = -\frac{1}{2}(b_{ij} + \mathbf{i}g_{ij}) & \text{if } k = i \\ \frac{1}{2\mathbf{i}}Y_{ij}^H = -\frac{1}{2}(b_{ij} - \mathbf{i}g_{ij}) & \text{if } k = j \\ 0 & \text{if } k \notin \{i, j\}. \end{cases}$$

Hence, for each line  $(j, k) \in E$ , the relevant angles for A2 are those of  $[C_0]_{jk}$  and

$$\begin{aligned} [\Phi_j]_{jk} &= -\frac{1}{2}(g_{jk} - \mathbf{i}b_{jk}) \\ [\Phi_k]_{jk} &= -\frac{1}{2}(g_{jk} + \mathbf{i}b_{jk}) \\ [\Psi_j]_{jk} &= -\frac{1}{2}(b_{jk} + \mathbf{i}g_{jk}) \\ [\Psi_k]_{jk} &= -\frac{1}{2}(b_{jk} - \mathbf{i}g_{jk}) \end{aligned}$$

as well as the angles of  $-[\Phi_j]_{jk}, -[\Phi_k]_{jk}$  and  $-\Psi_j]_{jk}, -[\Psi_k]_{jk}$ . These quantities are shown in Fig. 1 with their magnitudes normalized to a common value and explained in the caption of the figure.

Condition A2 applied to OPF (16) takes the following form (see Fig. 1):

A2': For each link  $(j, k) \in E$ , there is a line in the complex plane through the origin such that  $[C_0]_{jk}$  as well as those  $\pm[\Phi_i]_{jk}$  and  $\pm[\Psi_i]_{jk}$  corresponding to *finite* lower or upper bounds on  $(p_i, q_i)$ , for  $i = j, k$ , are all on one side of the line, possibly on the line itself.

Let  $C^{\text{opt}}$  and  $C^{\text{socp}}$  denote the optimal values of OPF (2) and OPF-socp (7), respectively.

*Corollary 3:* Suppose  $G$  is a tree and A2' holds.

- 1)  $C^{\text{opt}} = C^{\text{socp}}$ . Moreover, an optimal solution  $V^{\text{opt}}$  of OPF (2) can be recovered from every optimal solution  $W_G^{\text{socp}}$  of OPF-socp (7).
- 2) If, in addition, A1 holds, then OPF-socp (7) is exact.

It is clear from Fig. 1 that condition A2' cannot be satisfied if there is a line where the real and reactive power injections at both ends are both lower and upper bounded (eight combinations as shown in the figure). A2' requires that some of them be unconstrained even though in practice they are always bounded. It should be interpreted as requiring that the optimal solutions obtained by ignoring these bounds turn out to satisfy these bounds. This is generally different from solving the optimization *with* these constraints but requiring that they be inactive (strictly within these bounds) at optimality, unless the cost function is strictly convex. The result proved in [27] also includes constraints on real branch power flows and line losses. Corollary 3 includes several sufficient conditions in the literature for exact relaxation as special cases; see the caption of Fig. 1.

Corollary 3 also implies a result first proved in [16], using a different technique, that SOCP relaxation is exact in BFM for radial networks when there are no lower bounds on power injections  $s_j$ . The argument in [16] is generalized in [17, Part I] to allow convex objective functions, shunt elements, and line limits in terms of upper bounds on  $\ell_{jk}$ . Assume

A3: The cost function  $C(x)$  is convex, strictly increasing in  $\ell$ , nondecreasing in  $s = (p, q)$ , and independent of branch flows  $S = (P, Q)$ .

A4: For  $j \in N^+$ ,  $s_j = -\infty - \mathbf{i}\infty$ .

Popular cost functions in the literature include active power loss over the network or active power generations, both of which satisfy A3. The next result is proved in [16] and [17].

*Theorem 4:* Suppose  $\tilde{G}$  is a tree and A3–A4 hold. Then OPF-socp (13) is exact.

*Remark 2:* If the cost function  $C(x)$  in A3 is only nondecreasing, rather than strictly increasing, in  $\ell$ , then A3–A4 still guarantee that all optimal solutions of OPF (10) are (i.e., can be mapped to) optimal solutions of OPF-socp (13), but OPF-socp may have an optimal solution that maintains strict inequalities in (11c) and, hence, is infeasible for OPF. Even though OPF-socp is not exact in this case, the proof of Theorem 4 constructs from it an optimal solution of OPF. (See the proof in the arXiv version of this paper.)

## B. Voltage Upper Bounds

While type A conditions require that some power injection constraints not be binding, type B conditions require non-binding voltage upper bounds. They are proved in [32]–[35] using BFM.

For radial networks, the model originally proposed in [18] and [19], which is (11) with the inequalities in (11c) replaced by equalities, is exact. This is because the cycle condition (12) is always satisfied since the reduced incidence matrix  $B$  is  $n \times n$  and invertible for radial networks. Following [35], we adopt the graph orientation where every link points *towards* node 0. Then (11) for a radial network reduces to

$$S_{jk} = \sum_{i:i \rightarrow j} (S_{ij} - z_{ij}\ell_{ij}) + s_j, \quad j \in N^+ \quad (17a)$$

$$v_j - v_k = 2\text{Re}(z_{jk}^H S_{jk}) - |z_{jk}|^2 \ell_{jk}, \quad j \rightarrow k \in \tilde{E} \quad (17b)$$

$$v_j \ell_{jk} \geq |S_{jk}|^2, \quad j \rightarrow k \in \tilde{E} \quad (17c)$$

where  $v_0$  is given and in (17a),  $k$  denotes the node on the unique path from node  $j$  to node 0. The boundary condition is  $S_{jk} := 0$  when  $j = 0$  in (17a) and  $S_{ij} = 0, \ell_{ij} = 0$  when  $j$  is a leaf node.<sup>3</sup>

As before, the voltage magnitudes must satisfy

$$\underline{v}_j \leq v_j \leq \bar{v}_j, \quad j \in N. \quad (18a)$$

We allow more general constraints on the power injections: they can be in an *arbitrary* set  $\mathbb{S}_j$  that is bounded above

$$s_j \in \mathbb{S}_j \subseteq \{s_j \in \mathbb{C} | s_j \leq \bar{s}_j\}, \quad j \in N \quad (18b)$$

<sup>3</sup>A node  $j \in N$  is a *leaf node* if there is no  $i$  such that  $i \rightarrow j \in \tilde{E}$ .

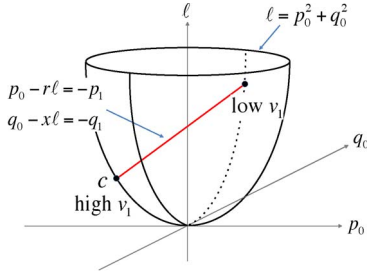


Fig. 2. Feasible set of OPF for a two-bus network without any constraint. It consists of the (two) points of intersection of the line with the convex surface (without the interior) and, hence, is nonconvex. SOCP relaxation includes the interior of the convex surface and enlarges the feasible set to the line segment joining these two points. If the cost function  $C$  is increasing in  $\ell$  or  $(p_0, q_0)$ , then the optimal point over the SOCP feasible set (line segment) is the lower feasible point  $c$  and, hence, the relaxation is exact. No constraint on  $\ell$  or  $(p_0, q_0)$  will destroy exactness as long as the resulting feasible set contains  $c$ .

for some given  $\bar{s}_j, j \in N$ .<sup>4</sup> Then the SOCP relaxation is

OPF-socp :

$$\min_x C(x) \text{ subject to (17), (18).} \quad (19)$$

OPF-socp (19) is exact if every optimal solution  $x^{\text{socp}}$  attains equality in (17c). In that case, an optimal solution of BFM (10) can be uniquely recovered from  $x^{\text{socp}}$ .

We make two comments on  $\mathbb{S}_j$  in (18b). First,  $\mathbb{S}_j$  need not be convex nor even connected for convex relaxations to be exact. They (only) need to be convex to be efficiently computable. Second, such a general constraint on  $s$  is useful in many applications. It includes the case where  $s_j$  are subject to simple box constraints, but also allows constraints of the form  $|s_j|^2 \leq a, |\angle s_j| \leq \phi_j$  that is useful for volt/var control [47], or  $q_j \in \{0, a\}$  for capacitor configurations.

**Geometric Insight:** To motivate condition B2 below, we first explain a simple intuition using a two-bus network on why relaxing voltage upper bounds guarantees exact SOCP relaxation. Consider bus 0 and bus 1 connected by a line with impedance  $z := r + ix$ . Suppose without loss of generality that  $v_0 = 1$  p.u. Eliminating  $S_{01} = s_0$  from (17), the model reduces to (dropping the subscript on  $\ell_{01}$ )

$$p_0 - r\ell = -p_1, \quad q_0 - x\ell = -q_1, \quad p_0^2 + q_0^2 = \ell \quad (20)$$

and

$$v_1 - v_0 = 2(rp_0 + xq_0) - |z|^2\ell. \quad (21)$$

Suppose  $s_1$  is given (e.g., a constant power load). Then the variables are  $(\ell, v_1, p_0, q_0)$  and the feasible set consists of solutions of (20) and (21) subject to additional constraints on  $(\ell, v_1, p_0, q_0)$ . The case without any constraint is instructive and shown in Fig. 2 (see explanation in the caption). The point  $c$  in the figure corresponds to a power flow solution with a large  $v_1$  (normal operation) whereas the other intersection corresponds to a solution with a small  $v_1$  (fault condition). As explained in the caption, SOCP relaxation is exact if there is no

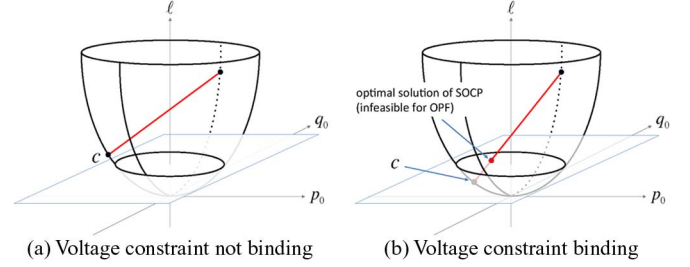


Fig. 3. Impact of voltage upper bound  $\bar{v}_1$  on exactness. (a) When  $\bar{v}_1$  (corresponding to a lower bound on  $\ell$ ) is not binding, the power flow solution  $c$  is in the feasible set of SOCP and, hence, the relaxation is exact. (b) When  $\bar{v}_1$  excludes  $c$  from the feasible set of SOCP, the optimal solution is infeasible for OPF and the relaxation is not exact.

voltage constraint and as long as constraints on  $(\ell, p_0, q_0)$  do not remove the high-voltage solution  $c$ . Only when the system is stressed so much that  $c$  becomes infeasible will relaxation lose exactness. This agrees with the conventional wisdom that power systems under normal operations are well behaved.

Consider now the voltage constraint  $\underline{v}_1 \leq v_1 \leq \bar{v}_1$ . Substituting (20) into (21), we obtain

$$v_1 = (1 + rp_1 + xq_1) - |z|^2\ell$$

translating the constraint on  $v_1$  into a box constraint on  $\ell$

$$\frac{1}{|z|^2}(rp_1 + xq_1 + 1 - \bar{v}_1) \leq \ell \leq \frac{1}{|z|^2}(rp_1 + xq_1 + 1 - \underline{v}_1).$$

Fig. 2 shows that the lower bound  $\underline{v}_1$  (corresponding to an upper bound on  $\ell$ ) does not affect the exactness of SOCP relaxation. The effect of upper bound  $\bar{v}_1$  (corresponding to a lower bound on  $\ell$ ) is illustrated in Fig. 3. As explained in the caption of the figure, SOCP relaxation is exact if the upper bound  $\bar{v}_1$  does not exclude the high-voltage solution  $c$  and is not exact otherwise.

For a general radial network, recall from [25, Sec. VI] the linear approximation of BFM for radial networks obtained by setting  $\ell_{jk} = 0$  in (17): for each  $s$

$$S_{jk}^{\text{lin}}(s) = \sum_{i \in \mathbb{T}_j} s_i \quad (22a)$$

$$v_j^{\text{lin}}(s) = v_0 + 2 \sum_{(i,k) \in \mathbb{P}_j} \text{Re}(z_{ik}^H S_{ik}^{\text{lin}}(s)) \quad (22b)$$

where  $\mathbb{T}_j$  denotes the subtree at node  $j$ , including  $j$ , and  $\mathbb{P}_j$  denotes the set of links on the unique path from  $j$  to 0. The key property we will use is from [25, Lemma 13 and Remark 9]:

$$S_{jk} \leq S_{jk}^{\text{lin}}(s) \text{ and } v_j \leq v_j^{\text{lin}}(s). \quad (23)$$

Define the  $2 \times 2$  matrix function

$$A_{jk}(S_{jk}, v_j) := I - \frac{2}{v_j} z_{jk}(S_{jk})^T \quad (24)$$

where  $z_{jk} := [r_{jk} \ x_{jk}]^T$  is the line impedance and  $S_{jk} := [P_{jk} \ Q_{jk}]^T$  is the branch power flows, both taken as real vectors so that  $z_{jk}(S_{jk})^T$  is a  $2 \times 2$  matrix with a rank less or equal to 1. The matrices  $A_{jk}(S_{jk}, v_j)$  describe how changes

<sup>4</sup>We assume here that  $s_0$  is unconstrained, and since  $V_0 := 1 \angle 0^\circ$  p.u., the constraints (18) involve only  $j$  in  $N$ , not  $N^+$ .

in branch power flows propagate towards the root node 0; see comments below. Evaluate the Jacobian matrix  $A_{jk}(S_{jk}, v_j)$  at the boundary values

$$\begin{aligned} \underline{A}_{jk} &:= A_{jk} \left( [S_{jk}^{\text{lin}}(\bar{s})]^+, v_j \right) \\ &:= I - \frac{2}{v_j} z_{jk} \left( [S_{jk}^{\text{lin}}(\bar{s})]^+ \right)^T. \end{aligned} \quad (25)$$

Here  $([a]^+)^T$  is the row vector  $[[a_1]^+ \ a_2]^+$  with  $[a_j]^+ := \max\{0, a_j\}$ .

For a radial network, for  $j \neq 0$ , every link  $j \rightarrow k$  identifies a unique node  $k$  and, therefore, to simplify notation, we refer to a link interchangeably by  $(j, k)$  or  $j$  and use  $A_j, \underline{A}_j, z_j$ , etc. in place of  $A_{jk}, \underline{A}_{jk}, z_{jk}$ , etc., respectively. Assume

- B1: The cost function is  $C(x) := \sum_{j=0}^n C_j(\text{Res}_j)$  with  $C_0$  strictly increasing. There is no constraint on  $s_0$ .  
 B2: The set  $\mathbb{S}_j$  of injections satisfies  $v_j^{\text{lin}}(s) \leq \bar{v}_j$ ,  $j \in N$ , where  $v_j^{\text{lin}}(s)$  is given by (22).  
 B3: For each leaf node  $j \in N$ , let the unique path from  $j$  to 0 have  $k$  links and be denoted by  $\mathbb{P}_j := ((i_k, i_{k-1}), \dots, (i_1, i_0))$  with  $i_k = j$  and  $i_0 = 0$ . Then  $\underline{A}_{i_t} \cdots \underline{A}_{i_{t'}} z_{i_{t'+1}} > 0$  for all  $1 \leq t \leq t' < k$ .

The following result is proved in [35].

**Theorem 5:** Suppose  $\tilde{G}$  is a tree and B1–B3 hold. Then OPF-socp (19) is exact.

We now comment on the conditions B1–B3. B1 requires that the cost functions  $C_j$  depend only on the injections  $s_j$ . For instance, if  $C_j(\text{Res}_j) = p_j$ , then the cost is total active power loss over the network. It also requires that  $C_0$  be strictly increasing but makes no assumption on  $C_j, j > 0$ . Common cost functions, such as line loss or generation cost, usually satisfy B1. If  $C_0$  is only nondecreasing, rather than strictly increasing, in  $p_0$ , then B1–B3 still guarantee that all optimal solutions of OPF (10) are (effectively) optimal for OPF-socp (19), but OPF-socp may not be exact, i.e., it may have an optimal solution that maintains strict inequalities in (17c). In this case, the proof of Theorem 5 can construct from it another optimal solution that attains equalities in (17c).

B2 is affine in the injections  $s := (p, q)$ . It enforces the upper bounds on voltage magnitudes because of (23).

B3 has a simple interpretation: the power flows  $S_{jk}$  on all branches should move in the same direction. Specifically, given a marginal change in the complex power on line  $j \rightarrow k$ , the  $2 \times 2$  matrix  $\underline{A}_{jk}$  is (a lower bound on) the Jacobian and describes the effect of this marginal change on the complex power on the line immediately upstream from line  $j \rightarrow k$ . The product of  $\underline{A}_i$  in B3 propagates this effect upstream toward the root. B3 requires that a small change, positive or negative, in the power flow on a line affects *all* upstream branch powers in the same direction. This seems to hold with a significant margin in practice; see [35] for examples from real systems.

Theorem 5 unifies and generalizes some earlier results in [32]–[34]. The sufficient conditions in these papers have the following simple and practical interpretation: OPF-socp is exact provided either

- there are no reverse power flows in the network, or
- if the  $r/x$  ratios on all lines are equal, or

- if the  $r/x$  ratios increase in the downstream direction from the substation (node 0) to the leaves, then there are no reverse real power flows, or
- if the  $r/x$  ratios decrease in the downstream direction, then there are no reverse reactive power flows.

The exactness of SOCP relaxation does not require convexity, i.e., the cost  $C(x) = \sum_{j=0}^n C_j(\text{Res}_j)$  need not be a convex function and the injection regions  $\mathbb{S}_j$  need not be convex sets. Convexity allows polynomial-time computation. Moreover, when it is convex, the exactness of SOCP relaxation also implies the uniqueness of the optimal solution, as the following result from [35] shows.

**Theorem 6:** Suppose  $\tilde{G}$  is a tree. Suppose the costs  $C_j$ ,  $j = 0, \dots, n$ , are convex functions and the injection regions  $\mathbb{S}_j$ ,  $j = 1, \dots, n$ , are convex sets. If the relaxation OPF-socp (19) is exact, then its optimal solution is unique.

Consider the model of [18] for radial networks, which is (17) with the inequalities in (17c) replaced by equalities. Let  $\mathbb{X}$  denote an equivalent feasible set of OPF,<sup>5</sup> i.e., those  $x \in \mathbb{R}^{3(m+n+1)}$  that satisfy (17), (18), and attain equalities in (17c). The proof of Theorem 6 reveals that, for radial networks, the feasible set  $\mathbb{X}$  has a “hollow” interior.

**Corollary 7:** Suppose  $\tilde{G}$  is a tree. If  $\hat{x}$  and  $\tilde{x}$  are distinct solutions in  $\mathbb{X}$ , then no convex combination of  $\hat{x}$  and  $\tilde{x}$  can be in  $\mathbb{X}$ . In particular,  $\mathbb{X}$  is nonconvex.

This property is illustrated vividly in several numerical examples for mesh networks in [48]–[51].

### C. Angle Differences

The sufficient conditions in [29], [36], and [37] require that the voltage angle difference across each line be small. We explain the intuition using a result in [36] for an OPF problem where  $|V_j|$  are fixed for all  $j \in N^+$  and reactive powers are ignored. Under these assumptions, as long as the voltage angle difference is small, the power flow solutions form a locally convex surface that is the Pareto front of its relaxation. This implies that the relaxation is exact. This geometric picture is apparent in earlier work on the geometry of power flow solutions, see e.g., [48], and underlies the intuition that the dynamics of a power system are usually benign until it is pushed towards the boundary of its stability region. The geometric insight in Figs. 2 and 3 for BFM and later in this subsection for BIM says that, when it is far away from the boundary, the local convexity structure also facilitates exact relaxation. Reactive power is considered in [37, Theor. 1] with fixed  $|V_j|$  where, with an additional constraint on the lower bounds of reactive power injections that ensure these lower bounds are not tight, it is proved that if the original OPF problem is feasible, then its SDP relaxation is exact. The case of variable  $|V_j|$  without reactive power is considered in [36, Theor. 7] but the simple geometric structure is lost.

<sup>5</sup>There is a bijection between  $\mathbb{X}$  and the feasible set of OPF (10) [when (18b) is replaced by (9b)] [17], [25].

Recall that  $y_{jk} = g_{jk} - \mathbf{i}b_{jk}$  with  $g_{jk} > 0, b_{jk} > 0$ . Let  $V_j = |V_j|e^{\mathbf{i}\theta_j}$  and suppose  $|V_j|$  are given. Consider

$$\min_{p, P, \theta} C(p) \quad (26a)$$

$$\text{s.t. } \underline{p}_j \leq p_j \leq \bar{p}_j, \quad j \in N^+ \quad (26b)$$

$$\underline{\theta}_{jk} \leq \theta_{jk} \leq \bar{\theta}_{jk}, \quad (j, k) \in E \quad (26c)$$

$$p_j = \sum_{k: k \sim j} P_{jk}, \quad j \in N^+ \quad (26d)$$

$$P_{jk} = |V_j|^2 g_{jk} - |V_j| |V_k| g_{jk} \cos \theta_{jk} + |V_j| |V_k| b_{jk} \sin \theta_{jk} \quad (j, k) \in E \quad (26e)$$

where  $\theta_{jk} := \theta_j - \theta_k$  are the voltage angle differences across lines  $(j, k)$ .

We comment on the constraints on angles  $\theta_{jk}$  in (26). When the voltage magnitudes  $|V_i|$  are fixed, constraints on real power flows, branch currents, line losses, as well as stability constraints can all be represented in terms of  $\theta_{jk}$ . Indeed, a line flow constraint of the form  $|P_{jk}| \leq \bar{P}_{jk}$  becomes a constraint on  $\theta_{jk}$  using the expression for  $P_{jk}$  in (26e). A current constraint of the form  $|I_{jk}| \leq \bar{I}_{jk}$  is also a constraint on  $\theta_{jk}$  since  $|I_{jk}|^2 = |y_{jk}|(|V_j|^2 + |V_k|^2 - 2|V_j V_k| \cos \theta_{jk})$ . The line loss over  $(j, k) \in E$  is equal to  $P_{jk} + P_{kj}$  which is again a function of  $\theta_{jk}$ . Stability typically requires  $|\theta_{jk}|$  to stay within a small threshold. Therefore, given constraints on branch power or current flows, losses, and stability, appropriate bounds  $\underline{\theta}_{jk}, \bar{\theta}_{jk}$  can be determined in terms of these constraints, assuming  $|V_j|$  are fixed.

We can eliminate the branch flows  $P_{jk}$  and angles  $\theta_{jk}$  from (26). Since  $|V_j|, j \in N^+$  are fixed, we assume without loss of generality that  $|V_j| = 1$  p.u.. Define the injection region

$$\mathbb{P}_\theta := \left\{ p \in \mathbb{R}^n \mid p_j = \sum_{k: k \sim j} (g_{jk} - g_{jk} \cos \theta_{jk} + b_{jk} \sin \theta_{jk}), \right. \\ \left. \underline{\theta}_{jk} \leq \theta_{jk} \leq \bar{\theta}_{jk}, j \in N^+, (j, k) \in E \right\} \quad (27)$$

Let  $\mathbb{P}_p := \{p \in \mathbb{R}^n \mid \underline{p}_j \leq p_j \leq \bar{p}_j, j \in N\}$ . Then (26) is

OPF :

$$\min_p C(p) \text{ subject to } p \in \mathbb{P}_\theta \cap \mathbb{P}_p. \quad (28)$$

This problem is hard because the set  $\mathbb{P}_\theta$  is nonconvex. To avoid triviality, we assume OPF (28) is feasible. For a set  $A$ , let  $\text{conv} A$  denote the convex hull of  $A$ . Consider the following problem that relaxes the nonconvex feasible set  $\mathbb{P}_\theta \cap \mathbb{P}_p$  of (28) to a convex superset

OPF-socp :

$$\min_p C(p) \text{ subject to } p \in \text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p. \quad (29)$$

We will show below that (29) is indeed an SOCP. It is said to be *exact* if every optimal solution of (29) lies in  $\mathbb{P}_\theta \cap \mathbb{P}_p$  and is therefore also optimal for (28).

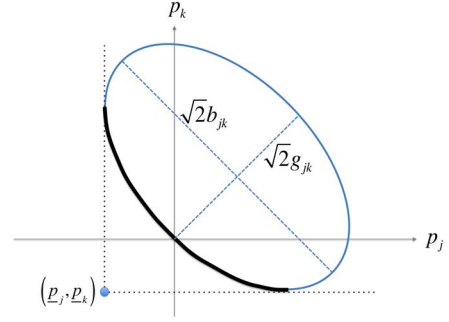


Fig. 4. Feasible set of OPF (28) for a two-bus network without any constraint when  $|V_j|$  are fixed and reactive powers are ignored. It is an ellipse without the interior; hence, nonconvex. OPF-socp (29) includes the interior of the ellipse and is hence convex. If the cost function  $C$  is strictly increasing in  $(p_j, p_k)$ , then the Pareto front of the SOCP feasible set will lie on the lower part of the ellipse  $\mathbb{O}(\mathbb{P}_\theta) = \mathbb{P}_\theta$  and, hence, OPF-socp is exact.

We say that a point  $x \in A \subseteq \mathbb{R}^n$  is a *Pareto optimal point* in  $A$  if there does not exist another  $x' \in A$  such that  $x' \leq x$  with at least one strictly smaller component  $x'_j < x_j$ . The *Pareto front of  $A$* , denoted by  $\mathbb{O}(A)$ , is the set of all Pareto optimal points in  $A$ . The significance of  $\mathbb{O}(A)$  is that, for any increasing function, its minimizer, if it exists, is necessarily in  $\mathbb{O}(A)$  whether  $A$  is convex or not. If  $A$  is convex, then  $x^{\text{opt}}$  is a Pareto optimal point in  $\mathbb{O}(A)$  if and only if there is a nonzero vector  $c := (c_1, \dots, c_n) \geq 0$  such that  $x^{\text{opt}}$  is a minimizer of  $c^T x$  over  $A$  [52, pp.179–180].

Assume

C1:  $C(p)$  is strictly increasing in each  $p_j$ .

C2: For all  $(j, k) \in E$ ,  $-\tan^{-1}(b_{jk}/g_{jk}) < \underline{\theta}_{jk} \leq \bar{\theta}_{jk} < \tan^{-1}(b_{jk}/g_{jk})$ .

The following result, proved in [36] and [37], says that (29) is exact, provided that  $\theta_{jk}$  are suitably bounded.

**Theorem 8:** Suppose  $G$  is a tree and C1–C2 hold.

1)  $\mathbb{P}_\theta \cap \mathbb{P}_p = \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$ .

2) The problem (29) is indeed an SOCP. Moreover, it is exact.

C1 is needed to ensure every optimal solution of OPF-socp (29) is optimal for OPF (28). If  $C(p)$  is nondecreasing but not strictly increasing in all  $p_j$ , then  $\mathbb{P}_\theta \cap \mathbb{P}_p \subseteq \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$  and OPF-socp may not be exact according to our definition. Even in that case, it is possible to recover an optimal solution of OPF from any optimal solution of OPF-socp.

Theorem 8 is illustrated in Figs. 4 and 5. As explained in the caption of Fig. 4, if there are no constraints, then SOCP relaxation (29) is exact under condition C1. It is clear from the figure that upper bounds on power injections do not affect exactness whereas lower bounds do. The purpose of condition C2 is to restrict the angle  $\theta_{jk}$  in order to eliminate the upper half of the ellipse from  $\mathbb{P}_\theta$ . As explained in the caption of Fig. 5, under C2,  $\mathbb{P}_\theta \cap \mathbb{P}_p = \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$  and, hence, the relaxation is exact. Otherwise, it may not.

When the network is not radial or  $|V_j|$  are not constants, then the feasible set can be much more complicated than ellipsoids [49]–[51]. Even in such settings, the Pareto fronts might still coincide, though the simple geometric picture is lost. See [48] for a numerical example on an Australian system or [24] for a three-bus mesh network.



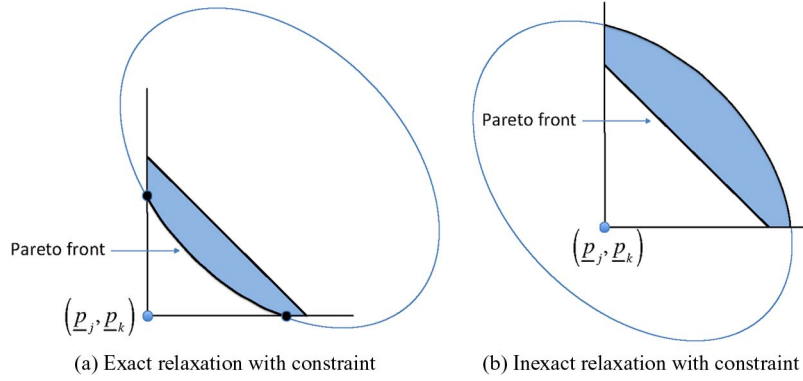


Fig. 5. With lower bounds  $\underline{p}$  on power injections, the feasible set of OPF-socp (29) is the shaded region. (a) When the feasible set of OPF (28) is restricted to the lower half of the ellipse (small  $|\theta_{jk}|$ ), the Pareto front remains on the ellipse itself  $\mathbb{P}_\theta \cap \mathbb{P}_p = \mathbb{Q}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$  and, hence, the relaxation is exact. (b) When the feasible set of OPF includes the upper half of the ellipse (large  $|\theta_{jk}|$ ), the Pareto front may not lie on the ellipse if  $\underline{p}$  is large, making the relaxation not exact.

#### D. Equivalence

Since BIM and BFM are equivalent, the results on exact SOCP relaxation and uniqueness of the optimal solution apply in both models. Recall the linear bijection  $g$  from BIM to BFM defined in [25, end of Sec. V] by  $x = g(W_G)$ , where

$$\begin{aligned} S_{jk} &:= y_{jk}^H \left( [W_G]_{jj} - [W_G]_{jk} \right), \quad j \rightarrow k \\ \ell_{jk} &:= |y_{jk}|^2 \left( [W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj} \right), \quad j \rightarrow k \\ v_j &:= [W_G]_{jj}, \quad j \in N^+ \\ s_j &:= \sum_{k: j \sim k} y_{jk}^H \left( [W_G]_{jj} - [W_G]_{jk} \right), \quad j \in N^+. \end{aligned}$$

The mapping  $g$  allows us to directly apply Theorem 6 to BIM. We summarize all of the results for type A and type B conditions for radial networks.<sup>6</sup>

**Theorem 9:** Suppose  $G$  and  $\tilde{G}$  are trees. Suppose conditions A1–A2', or A3–A4, or B1–B3 hold. Then

- 1) **BIM:** SOCP relaxation (7) is exact. Moreover, if  $C(W_G)$  is convex in  $([W_G]_{jj}, [W_G]_{jk})$ , then the optimal solution is unique.
- 2) **BFM:** SOCP relaxation (13) is exact. Moreover, if  $C(x) := \sum_j C_j(p_j)$  is convex in  $p$ , then the optimal solution is unique.

Since the SDP and the chordal relaxations are equivalent to the SOCP relaxation for radial networks, these results apply to SDP and chordal relaxations as well.

#### IV. MESH NETWORKS

In this section, we summarize a result of [17, Part II] on mesh networks with phase shifters and of [17, Part I], [39], [41] on dc networks when all voltages are non-negative.

To be able to recover an optimal solution of OPF from an optimal solution  $W_G^{\text{socp}}/x^{\text{socp}}$  of SOCP relaxation,  $W_G^{\text{socp}}/x^{\text{socp}}$

<sup>6</sup>To apply type C conditions to BFM, one needs to translate the angles  $\theta_{jk}$  to the BFM variables  $x := (S, \ell, v, s)$  through  $\beta_{jk}(x)$ , though this will introduce additional nonconvex constraints into OPF of the form  $\underline{\theta}_{jk} \leq \beta_{jk}(x) \leq \bar{\theta}_{jk}$ .

must satisfy a local condition and a global cycle condition [(4) for BIM and (12) for BFM]; see the definition of exactness in Section II. The conditions of Section III guarantee that every SOCP optimal solution will satisfy the local condition (i.e.,  $W_G^{\text{socp}}$  is  $2 \times 2$  psd rank-1 and  $x^{\text{socp}}$  attains equalities in (11c)), *whether the network is radial or mesh*, but do not guarantee that it satisfies the cycle condition. For radial networks, the cycle condition is vacuous and, therefore, the conditions of Section III are sufficient for SOCP relaxation to be exact. The result of [17, Part II] implies that these conditions are sufficient also for a mesh network that has tunable phase shifters at strategic locations.

Similar conditions also extend to dc networks where all variables are real and the voltages are assumed non-negative.

#### A. AC Networks With Phase Shifters

For BFM, the conditions of Section III guarantee that every optimal solution of OPF-socp (13) attains equalities in (11c) but may or may not satisfy the cycle condition (12). If it does, then it can be uniquely mapped to an optimal solution of OPF (10), according to [17, Theor. 2]. If it does not, then the solution is not physically implementable because it does not satisfy the power flow equations (Kirchhoff's laws). For a radial network, the reduced incidence matrix  $B$  in (12) is  $n \times n$  and invertible and, hence, every optimal solution of the SOCP relaxation that attains equalities in (11c) always satisfies the cycle condition [17, Theor. 4]. This is not the case for a mesh network where  $B$  is  $m \times n$  with  $m > n$ .

It is proved in [17, Part II], however, that if the network has tunable phase shifters, then any SOCP solution that attains equalities in (11c) becomes implementable even if the solution does not satisfy the cycle condition. This extends the sufficient conditions A1–A2', or A3–A4, or B1–B3, or C1–C2 from radial networks to this type of mesh network.

For BIM, the effect of phase shifter is equivalent to introducing a free variable  $\phi_c$  in (4) for each basis cycle  $c$  so that the cycle condition can always be satisfied for any  $W_G$ . The results presented here, however, start with a simple power flow model (30) for networks with phase shifters. This model makes transparent the spatial effect of phase shifters and its

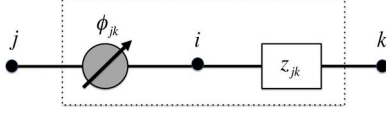


Fig. 6. Model of a phase shifter in line  $j \rightarrow k$ .

impact on the exactness of SOCP relaxation. It may be useful in other contexts, such as the design of a network of flexible ac transmission systems (FACTS) devices.

**BFM With Phase Shifters:** We consider an idealized phase shifter that only shifts the phase angles of the sending-end voltage and current across a line, and has no impedance nor limits on the shifted angles. Specifically consider an idealized phase shifter parametrized by  $\phi_{jk}$  across line  $j \rightarrow k$  as shown in Fig. 6. As before, let  $V_j$  denote the sending-end voltage at node  $j$ . Define  $I_{jk}$  to be the *sending-end* current leaving node  $j$  toward node  $k$ . Let  $i$  be the point between the phase shifter  $\phi_{jk}$  and line impedance  $z_{jk}$ . Let  $V_i$  and  $I_i$  be the voltage at  $i$  and the current from  $i$  to  $k$ , respectively. Then the effect of an idealized phase shifter, parametrized by  $\phi_{jk}$ , is summarized by the following modeling assumptions:

$$V_i = V_j e^{i\phi_{jk}} \text{ and } I_i = I_{jk} e^{i\phi_{jk}}.$$

The power transferred from nodes  $j$  to  $k$  is still (defined to be)  $S_{jk} := V_j I_{jk}^H$ , which is equal to the power  $V_i I_i^H$  from nodes  $i$  to  $k$  since the phase shifter is assumed to be lossless. Applying Ohm's law across  $z_{jk}$ , we define the *branch flow model with phase shifters* as the following set of equations:

$$\sum_{k:j \rightarrow k} S_{jk} = \sum_{i:i \rightarrow j} (S_{ij} - z_{ij} |I_{ij}|^2) + s_j, \quad j \in N^+ \quad (30a)$$

$$I_{jk} = y_{jk}(V_j - V_k e^{-i\phi_{jk}}), \quad j \rightarrow k \in \tilde{E} \quad (30b)$$

$$S_{jk} = V_j I_{jk}^H, \quad j \rightarrow k \in \tilde{E}. \quad (30c)$$

Without phase shifters ( $\phi_{jk} = 0$ ), (30) reduces to BFM (8). Let  $\tilde{x} := (S, I, V, s) \in \mathbb{C}^{2(m+n+1)}$  denote the variables in (30). Let  $x := (S, \ell, v, s) \in \mathbb{R}^{3(m+n+1)}$  denote the variables in SOCP relaxation (13). These variables are related through the mapping  $x = h(\tilde{x})$  where  $\ell_{jk} = |I_{jk}|^2$  and  $v_j = |V_j|^2$ . In particular, given any solution  $\tilde{x}$  of (30),  $x := h(\tilde{x})$  satisfies (11) with equalities in (11c).

**Cycle Condition:** If every line has a phase shifter, then the cycle condition changes from (12) to: given any  $x$  that satisfies (11) with equalities in (11c)

$$\exists(\theta, \phi) \in \mathbb{R}^{n+m} \text{ such that } B\theta = \beta(x) - \phi \pmod{2\pi}. \quad (31)$$

It is proved in [17, Part II] that, given any  $x$  that attains equalities in (11c), there always exists a  $\theta$  in  $(-\pi, \pi]^n$  and a  $\phi$  in  $(-\pi, \pi]^m$  that solve (31). Moreover, phase shifters are needed only on lines not in a spanning tree.

**Exact SOCP Relaxation:** Recall the OPF problem (10) where the feasible set  $\tilde{\mathbb{X}}$  without phase shifters is

$$\tilde{\mathbb{X}} := \{\tilde{x} | \tilde{x} \text{ satisfies (30) with } \phi = 0 \text{ and (9)}\}.$$

Phase shifters on every line enlarge the feasible set to

$$\bar{\mathbb{X}} := \{\tilde{x} | \tilde{x} \text{ satisfies (30) for some } \phi \text{ and (9)}\}.$$

Given any spanning tree  $T$  of  $\tilde{G}$ , let " $\phi \in T^\perp$ " be the shorthand for " $\phi_{jk} = 0$  for all  $(j, k) \in T$ ", i.e.,  $\phi$  involves only phase shifters in lines not in the spanning tree  $T$ . Fix any  $T$ . Define the feasible set when there are phase shifters only on lines outside  $T$

$$\bar{\mathbb{X}}_T := \{\tilde{x} | \tilde{x} \text{ satisfies (30) for some } \phi \in T^\perp \text{ and (9)}\}.$$

Clearly  $\tilde{\mathbb{X}} \subseteq \bar{\mathbb{X}}_T \subseteq \bar{\mathbb{X}}$ . Define the (modified) OPF problem where there is a phase shifter on every line

*OPF-ps :*

$$\min_{\tilde{x}, \phi} C(x) \text{ subject to } \tilde{x} \in \bar{\mathbb{X}}, \phi \in \mathbb{R}^m \quad (32)$$

and that where there are phase shifters only outside  $T$

*OPF-T :*

$$\min_{\tilde{x}, \phi} C(x) \text{ subject to } \tilde{x} \in \bar{\mathbb{X}}_T, \phi \in T^\perp. \quad (33)$$

Let  $C^{\text{opt}}$ ,  $C^{\text{ps}}$ , and  $C^T$  denote, respectively, the optimal values of OPF (10), OPF-ps (32), and OPF-T (33). Clearly,  $C^{\text{opt}} \geq C^T \geq C^{\text{ps}}$  since  $\tilde{\mathbb{X}} \subseteq \bar{\mathbb{X}}_T \subseteq \bar{\mathbb{X}}$ . Solving OPF (10), OPF-ps (32), or OPF-T (33) is difficult because their feasible sets are nonconvex.

Recall the following sets defined in [25] for networks without phase shifters

$$\mathbb{X}^+ := \{x | x \text{ satisfies (9) and (11)}\}$$

$$\mathbb{X}_{nc} := \{x | x \text{ satisfies (9) and (11) with equalities in (11c)}\}$$

$$\mathbb{X} := \{x | x \in \mathbb{X}_{nc} \text{ and satisfies the cycle condition (12)}\}.$$

Note that  $\mathbb{X}$  is defined by the cycle condition without phase shifters [ $\phi = 0$  in (31)]. As explained in [25, Theor. 9],  $\mathbb{X}$  is equivalent to the feasible set  $\tilde{\mathbb{X}}$  of OPF (10). Hence,  $\tilde{\mathbb{X}} \equiv \mathbb{X} \subseteq \mathbb{X}_{nc} \subseteq \mathbb{X}^+$ . A key result of [17, Part II] is

**Theorem 10:** Fix any spanning tree  $T$  of  $\tilde{G}$ . Then  $\bar{\mathbb{X}}_T \equiv \mathbb{X} \equiv \mathbb{X}_{nc}$ .

The implication of Theorem 10 is that, for a mesh network, when a solution of SOCP relaxation (13) attains equalities in (11c) (i.e., it is in  $\mathbb{X}_{nc}$ ), then it can be implemented with an appropriate setting of phase shifters even when the solution does not satisfy the cycle condition (12). Define the problem

*OPF-nc :*

$$\min_x C(x) \text{ subject to } x \in \mathbb{X}_{nc}. \quad (34)$$

Let  $C^{\text{nc}}$  and  $C^{\text{socp}}$  denote, respectively, the optimal values of OPF-nc (34) and OPF-socp (13). Theorem 10 then implies the following.

**Corollary 11:** Fix any spanning tree  $T$  of  $\tilde{G}$ . Then

- 1)  $\tilde{\mathbb{X}} \subseteq \bar{\mathbb{X}}_T = \bar{\mathbb{X}} \equiv \mathbb{X}_{nc} \subseteq \mathbb{X}^+$ .
- 2)  $C^{\text{opt}} \geq C^T = C^{\text{ps}} = C^{\text{nc}} \geq C^{\text{socp}}$ .

Hence, if an optimal solution  $x^{\text{socp}}$  of OPF-socp (13) attains equalities in (11c), then  $x^{\text{socp}}$  solves the problem OPF-nc (34). If it also satisfies the cycle condition (12), then  $x^{\text{socp}} \in \mathbb{X}$  and it can be mapped to a unique optimal of OPF (10). Otherwise,  $x^{\text{socp}}$  can be implemented through an appropriate phase-shifter setting  $\phi$  and it attains a cost that lower bounds the optimal cost of the original OPF without tunable phase shifters. Moreover, this benefit can be attained with phase shifters only outside an arbitrary spanning tree  $T$  of  $\tilde{G}$ . The result can help determine if a network with a given set of phase shifters can be convexified and, if not, where additional phase shifters are needed for convexification [17, Part II]. If the SOCP is exact, then phase shifters cannot further reduce the cost. This can help determine when phase shifters provide benefits to system operations.

Hence, phase shifters in strategic locations make a mesh network behave like a radial network as far as convex relaxation is concerned. The results of Section III then imply the following:

*Corollary 12:* Suppose conditions A1–A2', or A3–A4, or B1–B3, or C1–C2 hold. Then any optimal solution of OPF-socp (13) solves OPF-ps (32) and OPF- $T$  (33).

### B. DC Networks

In this subsection, we consider purely resistive dc networks, i.e., the impedance  $z_{jk} = r_{jk} = y_{jk}^{-1}$ , the power injections  $s_j = p_j$ , and the voltages  $V_j$  are real. We assume all voltage magnitudes are strictly positive. Formally:

D0: Replace (1b) and (11b) by  $0 < \underline{V}_j \leq V_j \leq \bar{V}_j$ ,  $j \in N^+$ , and replace (3b) by  $0 < \underline{V}_j^2 \leq [W_G]_{jj} \leq \bar{V}_j^2$ ,  $j \in N^+$ .

*Type A Conditions:* Condition D0 immediately implies that the cycle condition (12) in BFM is satisfied by every feasible  $x$  of OPF-socp (13), for

$$\begin{aligned} \beta_{jk}(x) &:= \angle(v_j - z_{jk}^H s_{jk}) \\ &= \angle\left(v_j - r_{jk} \left(r_{jk}^{-1} V_j (V_j - V_k)\right)\right) = 0^\circ. \end{aligned}$$

A3–A4 guarantee that any optimal solution of OPF-socp attains equalities in (11c) for general mesh networks. Hence, [25, Theor. 7] and Theorem 4 imply the following.

*Corollary 13:* Suppose A3–A4 and D0 hold. Then OPF-socp (13) is exact.

For BIM, an OPF as a QCQP has real and symmetric matrices in (16). Even though they satisfy condition A2', Corollary 3 is not applicable as its proof constructs a complex (not real)  $V$  from an optimal solution of OPF-socp. However, if there are no lower bounds on the power injections, then only  $\Phi_j$  are involved in the QCQP so all of their off-diagonal entries are non-positive. It is then observed in [39] that [46, Theor. 3.1] directly implies (without needing D0) the following.

*Corollary 14:* Suppose A1 and A4 hold. Then OPF-sdp (5) and OPF-socp (7) are exact.

*Type B Conditions:* The following result is proved in [41]. Consider:

B1': The cost function is  $C(x) := \sum_{j=0}^n C_j(\text{Res}_j)$  with  $C_j$  strictly increasing for all  $j \in N^+$ . There is no constraint on  $s_0$ .

B2':  $\bar{V}_1 = \bar{V}_2 = \dots = \bar{V}_n$ ;  $\mathbb{S}_j = [p_j, \bar{p}_j]$  with  $p_j < 0$ ,  $j \in N$ .

B2'':  $\bar{V}_j = \infty$  for  $j \in N$ .

*Theorem 15:* Suppose at least one of the following holds:

- B1, B2'' and D0; or
- B1', B2' and D0.

Then OPF-socp (7) with the additional constraints  $W_{jk} \geq 0$ ,  $(j, k) \in E$ , is exact. If, in addition, the problem is convex, then its optimal solution is unique.

It is possible to enforce B2'' by an affine constraint on the power injections, similar to (but different from) condition B2 for radial networks; see [41] for details. See also [53] for a result on the uniqueness of SOCP relaxation.

### C. General AC Networks

Unfortunately, no sufficient conditions for exact semidefinite relaxation for general mesh networks are yet known. There are type A conditions on power injections for exact relaxation only for special cases: a lossless cycle or lossless cycle with one chord [29], or a weakly cyclic network (where every line belongs to at most one cycle) of size 3 [54].

We close by mentioning three recent approaches for global optimization of OPF when the relaxations in this tutorial fail. First, higher-order semidefinite relaxations on the Lesser hierarchy for polynomial optimization [55] have been applied to solving OPF when SDP relaxation fails [56]–[59]. By going up the hierarchy, the relaxations become tighter and their solutions approach a global optimal of the original polynomial optimization [55], [60]. This, however, comes at the cost of significantly higher runtime. Techniques are proposed in [58] and [59] to reduce the problem sizes, e.g., by exploiting sparsity or adding redundant constraints [59], [61], [62] or applying higher-order relaxations only on (typically small) subnetworks where constraints are violated [58].

Second, a branch-and-bound algorithm is proposed in [63] where a lower bound is computed from the Lagrangian dual of OPF and the feasible set subdivision is based on rectangular or ellipsoidal bisection. The dual problem is solved using a subgradient algorithm. Each iteration of the subgradient algorithm requires minimizing the Lagrangian over the primal variables. This minimization is separable into two subproblems—one being a convex subproblem and the other having a nonconvex quadratic objective. The latter subproblem turns out to be a trust-region problem that has a closed-form solution. It is proved in [63] that the proposed algorithm converges to a global optimal. This method is extended in [64] to include more constraints and alternatively use SDP relaxation for lower bounding the cost.

Finally, a new approach is proposed in [65] based on convex quadratic relaxation of OPF in polar coordinates.

## V. CONCLUSION

We have summarized the main sufficient conditions for exact semidefinite relaxations of OPF as listed in Tables I and II. For radial networks, these conditions suggest that SOCP relaxation (and, hence, SDP and chordal relaxations) will likely be exact in practice. This is corroborated by significant numerical



experience. For mesh networks, they are applicable only for special cases: networks that have tunable phase shifters or dc networks where all variables are real and voltages are non-negative. Even though counterexamples exist where SDP/chordal relaxation is not exact for ac mesh networks, numerical experience suggests that SDP/chordal relaxation tends to be exact in many cases. Sufficient conditions that guarantee exact relaxation for ac mesh networks, however, remain elusive. The main difficulty is in designing relaxations of the cycle condition (4) or (12).

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